

ON THE EQUIVALENCE OF STANDARD AND NONSTANDARD HOMOLOGY THEORIES OF UNIFORM SPACES

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ABSTRACT. We introduce some nonstandard notions of homotopy and connectedness, and clarify the relation between standard and nonstandard ones. Uniform Vietoris homology satisfies the extremely strong form of the homotopy axiom in this context. Consequently, a precompact uniform space and its dense subspace are indistinguishable by their uniform Vietoris homology. In the preceding paper [5], we introduced a nonstandard homology of uniform spaces. We establish the equivalence of our homology and uniform Vietoris homology for precompact uniform spaces. Korppi [8] introduced a nonstandard homology theory of completely regular spaces with some nice properties. We prove that Korppi homology is equivalent to our homology for all fine uniform spaces. The excision and continuity axioms for our homology are also proved.

1. INTRODUCTION

1.1. Background. Let X and Y be any sets. For each relation R on $X \times Y$ the *Vietoris complex* $V(X, Y, R)$ is the simplicial set whose p -simplices are members (a_0, \dots, a_p) of X^{p+1} such that there exists a $b \in Y$ with $a_i R b$ ($0 \leq i \leq p$).

Let (X, A) be a topological pair. We denote by $\text{Cov}(X)$ the set of all open covers of X . Recall that $\mathcal{V} \subseteq \mathcal{P}(X)$ is called a *refinement* of $\mathcal{U} \subseteq \mathcal{P}(X)$ (denoted by $\mathcal{U} \preceq \mathcal{V}$) if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ with $V \subseteq U$. $(\text{Cov}(X), \preceq)$ is a directed set: \preceq is reflexive and transitive, and any $\mathcal{U}, \mathcal{V} \in \text{Cov}(X)$ have an upper bound $\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$. For $\mathcal{U} \in \text{Cov}(X)$ let $X_{\mathcal{U}} = V(X, \mathcal{U}, \in)$ and $A_{\mathcal{U}} = V(A, \mathcal{U}, \in)$. If $\mathcal{U} \preceq \mathcal{V}$, then $X_{\mathcal{V}}$ and $A_{\mathcal{V}}$ are simplicial subsets of $X_{\mathcal{U}}$ and $A_{\mathcal{U}}$, respectively. The *Vietoris homology* of (X, A) is defined as the inverse limit

$$\check{H}_{\bullet}(X, A; G) = \lim_{\mathcal{U} \in \text{Cov}(X)} H_{\bullet}(X_{\mathcal{U}}, A_{\mathcal{U}}; G),$$

where H_{\bullet} is the ordinary homology functor of simplicial pairs. Let $f : (X, A) \rightarrow (Y, B)$ be a continuous map. Let $\pi_{\mathcal{U}} : \check{H}_{\bullet}(X, A; G) \rightarrow H_{\bullet}(X_{\mathcal{U}}, A_{\mathcal{U}}; G)$ be the projection for $\mathcal{U} \in \text{Cov}(X)$. For each $\mathcal{V} \in \text{Cov}(Y)$, $f^{-1}\mathcal{V} \in \text{Cov}(X)$. f can be seen as a simplicial map from $(X_{f^{-1}\mathcal{V}}, A_{f^{-1}\mathcal{V}})$ to $(Y_{\mathcal{V}}, B_{\mathcal{V}})$. Let us define $f_{\mathcal{V}} = H_{\bullet}(f; G) \circ \pi_{f^{-1}\mathcal{V}} : \check{H}_{\bullet}(X, A; G) \rightarrow H_{\bullet}(Y_{\mathcal{V}}, B_{\mathcal{V}}; G)$. The following diagram commutes for all $\mathcal{V} \preceq \mathcal{W}$:

$$\begin{array}{ccc} & \check{H}_{\bullet}(X, A; G) & \\ f_{\mathcal{V}} \swarrow & & \searrow f_{\mathcal{W}} \\ H_{\bullet}(Y_{\mathcal{V}}, B_{\mathcal{V}}; G) & \xleftarrow{p_{\mathcal{W}\mathcal{V}}} & H_{\bullet}(Y_{\mathcal{W}}, B_{\mathcal{W}}; G) \end{array}$$

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By the universal property of $\check{H}_\bullet(Y, B; G)$, we have the homomorphism $\check{H}_\bullet(f; G) : \check{H}_\bullet(X, A; G) \rightarrow \check{H}_\bullet(Y, B; G)$. It is easy to verify that \check{H}_\bullet is a functor.

Intuitively, Vietoris homology is the homology of the Vietoris complex of the *ideal* infinitely fine open cover. McCord [9] developed a nonstandard homology of topological spaces by using nonstandard analysis. McCord homology is the homology of the Vietoris complex of the *real* infinitely fine open cover which consists of the monads of all standard points. Garavaglia [3] proved that these theories are equivalent for compact spaces.

Next, let (X, A) be a uniform pair. Let $\text{Cov}_u(X)$ denote the set of all uniform covers of X . Here $\mathcal{U} \subseteq \mathcal{P}(X)$ is called a *uniform cover* of X if there is an entourage U of X such that $\mathcal{U} \preceq \{U[x] \mid x \in X\}$, where $U[x] = \{y \in X \mid (x, y) \in U\}$. If $\mathcal{U} \preceq \{U[x] \mid x \in X\}$ and $\mathcal{V} \preceq \{V[x] \mid x \in X\}$, then $\mathcal{U} \vee \mathcal{V} \preceq \{(U \cap V)[x] \mid x \in X\}$. Hence $(\text{Cov}_u(X), \preceq)$ is a directed set. The *uniform Vietoris homology* of (X, A) is the inverse limit

$$\check{H}_\bullet^u(X, A; G) = \lim_{\mathcal{U} \in \text{Cov}_u(X)} H_\bullet(X_{\mathcal{U}}, A_{\mathcal{U}}; G).$$

The morphism part of \check{H}_\bullet^u can be defined as similar to that of \check{H}_\bullet . Note that if $f : (X, A) \rightarrow (Y, B)$ is uniformly continuous, then for each $\mathcal{V} \in \text{Cov}_u(Y)$, $f^{-1}\mathcal{V} \in \text{Cov}_u(X)$. Thus we have the functor \check{H}_\bullet^u .

Uniform Vietoris homology is the homology of the Vietoris complex of the *ideal* infinitely fine uniform cover. In the preceding paper [5], we introduced a nonstandard homology of uniform spaces, called μ -homology, as a modification of McCord's theory. μ -homology is the homology of the Vietoris complex of the *real* infinitely fine uniform cover which consists of the monads of all standard and nonstandard points. As we proved later, in the precompact case, μ -homology agrees with uniform Vietoris homology.

1.2. Summary. In Section 2, we introduce some nonstandard notions of homotopy and connectedness, and clarify the relation between standard and nonstandard ones. Uniform Vietoris homotopy, a standard notion of homotopy, is also introduced here. In Section 3, we first prove the equivalence of μ -homology and uniform Vietoris homology in the compact case. Next, we show that uniform Vietoris homology satisfies the nonstandard homotopy axiom for all precompact uniform spaces. We have as a consequence that a precompact uniform space and its dense subspace are indistinguishable by their uniform Vietoris homology. It follows that μ -homology and uniform Vietoris homology are equivalent also in the precompact case. In Section 4, we prove that Korppi homology is equivalent to μ -homology for all fine uniform spaces. The excision and continuity axioms for μ -homology are also proved.

1.3. Notations. A bit of nonstandard topology is required. We refer to the books [12] and [13]. As in [5] we use a standard universe \mathbb{U} and a nonstandard universe $^*\mathbb{U}$. The standard universe \mathbb{U} is assumed to satisfy sufficiently many axioms of ZFC and have all standard mathematical objects we consider. The nonstandard universe $^*\mathbb{U}$ is assumed to be a $|\mathbb{U}|^+$ -saturated elementary extension of \mathbb{U} . We denote by $X \mapsto {}^*X$ the embedding $\mathbb{U} \hookrightarrow {}^*\mathbb{U}$. We omit the star of *X when X plays the role of an atomic object (such as a number and a point of a space). For a standard concept X on \mathbb{U} definable in the language of \mathbb{U} , the nonstandard concept on ${}^*\mathbb{U}$ definable by the same formula as X is often called *internal* X , *hyper* X , *X , etc.

Let I denote the closed unit interval $[0, 1]$. We denote by $\mu(\cdot)$ the monad of a topological or uniform space and by \approx the infinite closeness relation of a uniform space. Let H_\bullet^μ denote the μ -homology functor. We also use the following symbols:

- TPair**: the category of topological pairs with continuous maps;
- CRPair**: the full subcategory of **UPair** whose objects are completely regular;
- UPair**: the category of uniform pairs with uniformly continuous maps;
- pUPair**: the full subcategory of **UPair** whose objects are precompact;
- C_h**: the homotopy category of a given category **C**.

2. NONSTANDARD HOMOTOPY AND CONNECTEDNESS

2.1. S-continuity, shadow and preshadow.

Definition 2.1. Let X and Y be uniform spaces. An internal map $f : *X \rightarrow *Y$ is said to be *S-continuous* at $x \in *X$ if $f(x) \approx f(y)$ holds for all $y \in *X$ with $y \approx x$, or more simply, if $f(\mu(x)) \subseteq \mu(f(x))$ holds.

Proposition 2.2. *For any S-continuous maps $f : *X \rightarrow *Y$ and $g : *Y \rightarrow *Z$ the composition $gf : *X \rightarrow *Z$ is also S-continuous.*

Proof. For any $x \in *X$ we have that $gf(\mu(x)) \subseteq g(\mu(f(x))) \subseteq \mu(gf(x))$. \square

For each full subcategory **C** of **UPair**, we denote by **C_S** the category whose objects are the same as **C** and whose morphisms are S-continuous maps between (the nonstandard extensions of) objects. By sending uniformly continuous maps to the nonstandard extensions, **C** can be embedded into **C_S**. Note that this embedding is faithful but not full in general.

Example 2.3. The closed n -ball \mathbb{B}^n and the open n -ball $\mathbb{B}^n \setminus \mathbb{S}^{n-1}$ are S-homeomorphic. For any real number $0 < r < 1$, $[r, 1]$ and $[r, 1)$ are equipotent. By transfer, for a hyperreal number $r \approx 1$, there exists an internal bijection $R : * [r, 1] \rightarrow * [r, 1)$. Let us define internal maps $f, g : \mathbb{B}^n \rightleftarrows \mathbb{B}^n \setminus \mathbb{S}^{n-1}$ as follows:

$$f(x) = \begin{cases} x, & 0 \leq \|x\| < r, \\ R(\|x\|) \frac{x}{\|x\|}, & r \leq \|x\| \leq 1, \end{cases}$$

$$g(x) = \begin{cases} x, & 0 \leq \|x\| < r, \\ R^{-1}(\|x\|) \frac{x}{\|x\|}, & r \leq \|x\| < 1. \end{cases}$$

They are S-continuous, $fg = \text{id}_{\mathbb{B}^n}$ and $gf = \text{id}_{\mathbb{B}^n \setminus \mathbb{S}^{n-1}}$.

Definition 2.4. Let X and Y be uniform spaces. An internal map $f : *X \rightarrow *Y$ is said to be $\varepsilon\delta$ -continuous at $x \in *X$ if for each entourage V of Y there is an entourage U of X such that $f(*U[x]) \subseteq *V[f(x)]$ holds. f is said to be *uniformly $\varepsilon\delta$ -continuous* if for each entourage V of Y there is an entourage U of X such that $f(*U[x]) \subseteq *V[f(x)]$ holds for all $x \in *X$.

The following two are well-known ([13, Theorem 8.4.22 and 8.4.23]). They state that S-continuity is equivalent to $\varepsilon\delta$ -continuity. In that sense, S-continuity is Standard-continuity.

Lemma 2.5. *Let X and Y be uniform spaces. Let $f : *X \rightarrow *Y$ be an internal map and $x \in *X$. The following are equivalent:*

- (1) f is S -continuous at x ;
- (2) f is $\varepsilon\delta$ -continuous at x .

Proof. (1) \Rightarrow (2): Let V be an entourage of Y . Consider the following set of internal entourages of *X :

$$\mathcal{A} = \{ U \mid f(U[x]) \subseteq {}^*V[f(x)] \}.$$

Since f is S -continuous at x , \mathcal{A} is internal and has all infinitely small internal entourages of *X . By the overspill principle (a special case of the saturation principle), there is an entourage U of X with ${}^*U \in \mathcal{A}$. Therefore $f({}^*U[x]) \subseteq {}^*V[f(x)]$.

- (2) \Rightarrow (1): For each entourage V of Y there is an entourage U of X such that $f({}^*U[x]) \subseteq {}^*V[f(x)]$ holds. Since $\mu(x) \subseteq {}^*U[x]$, we have $f(\mu(x)) \subseteq {}^*V[f(x)]$ for all V and hence $f(\mu(x)) \subseteq \mu(f(x))$. □

Lemma 2.6. *Let X and Y be uniform spaces. Let $f : {}^*X \rightarrow {}^*Y$ be an internal map. The following are equivalent:*

- (1) f is S -continuous;
- (2) f is uniformly $\varepsilon\delta$ -continuous.

Proof. (1) \Rightarrow (2): Let V be an entourage of Y . Consider the following set of internal entourages of *X :

$$\mathcal{A} = \{ U \mid f(U[x]) \subseteq {}^*V[f(x)] \text{ for all } x \in {}^*X \}.$$

\mathcal{A} is internal and has all infinitely small internal entourages of *X . By the overspill principle, there is an entourage U of X with ${}^*U \in \mathcal{A}$. Therefore $f({}^*U[x]) \subseteq {}^*V[f(x)]$ holds for all $x \in {}^*X$.

- (2) \Rightarrow (1): In particular, f is $\varepsilon\delta$ -continuous on *X . By Lemma 2.5, f is S -continuous on *X . □

Definition 2.7. Let X be a topological space. A point $x \in {}^*X$ is said to be *nearstandard* if there exists a $\xi \in X$ with $x \in \mu(\xi)$. Let $\text{ns}(X)$ denote the set of all nearstandard points of *X .

The following lemma is also well-known. This can be used for proving Arzelà-Ascoli theorem (see [13, Theorem 8.4.43]).

Lemma 2.8 (Shadow Lemma). *Let (X, A) and (Y, B) be uniform pairs. Let $f : {}^*(X, A) \rightarrow {}^*(Y, B)$ be an internal map. If f is S -continuous on X , $f({}^*X) \subseteq \text{ns}(Y)$ and $f({}^*A) \subseteq \text{ns}(B)$, then there exists a continuous map ${}^\circ f : (X, A) \rightarrow (Y, B)$ (called a shadow of f) which satisfies ${}^\circ f(x) \approx f(x)$ on X . If, in addition, X is compact, then ${}^\circ f$ is uniformly continuous and satisfies ${}^*({}^\circ f)(x) \approx f(x)$ on *X .*

Proof. Since $f : {}^*(X, A) \rightarrow (\text{ns}(Y), \text{ns}(B))$, one can define a map ${}^\circ f : (X, A) \rightarrow (Y, B)$ satisfying ${}^\circ f(x) \approx f(x)$ on X . Let $x \in X$ and V an entourage of Y . There exists an entourage $\sqrt[3]{V}$ of Y with $\sqrt[3]{V}^3 \subseteq V$. Since ${}^\circ f(x) \approx f(x)$, we have $({}^\circ f(x), f(x)) \in \sqrt[3]{V}$. By Lemma 2.5 there is an entourage U of X such that for any $y \in {}^*X$ if $(x, y) \in {}^*U$ then $(f(x), f(y)) \in \sqrt[3]{V}$. If, in addition, $y \in X$, then by $(f(y), {}^\circ f(y)) \in \sqrt[3]{V}$ and transfer, we obtain $({}^\circ f(x), {}^\circ f(y)) \in V$. ${}^\circ f$ is continuous at x .

Suppose X is compact. Let $x \in {}^*X$. By [12, Theorem 4.1.13], x is nearstandard. There exists a $\xi \in X$ with $x \approx \xi$. By transfer and Lemma 2.5, ${}^*(\circ f)$ is S-continuous at ξ . We have ${}^*(\circ f(x)) \approx \circ f(\xi) \approx f(\xi) \approx f(x)$. \square

Definition 2.9. Let X be a uniform space. A point $x \in {}^*X$ is said to be *pre-nearstandard* if for each entourage U of X there is a $\xi \in X$ with $x \in {}^*U[\xi]$. Let $\text{pns}(X)$ denote the set of all pre-nearstandard points of *X .

Definition 2.10. Let X and Y be uniform spaces. A map $f : X \rightarrow Y$ is said to be *V-continuous* at $x \in X$, where V is an entourage of Y , if there is an entourage U of X such that $f(U[x]) \subseteq V[f(x)]$. f is said to be *uniformly V-continuous*, where V is an entourage of Y , if there is an entourage U of X such that $f(U[x]) \subseteq V[f(x)]$ for all $x \in X$.

Lemma 2.11 (Preshadow Lemma). *Let (X, A) and (Y, B) be uniform pairs. Let $f : {}^*(X, A) \rightarrow {}^*(Y, B)$ be an internal map. If f is S-continuous on X , $f({}^*X) \subseteq \text{pns}(Y)$ and $f({}^*A) \subseteq \text{pns}(B)$, then for each entourage V of Y , there exists a V -continuous map ${}^V f : (X, A) \rightarrow (Y, B)$ (called a V -preshadow of f) which satisfies $({}^V f(x), f(x)) \in {}^*V$ on X . Moreover, if f is S-continuous on *X , then ${}^V f$ is uniformly V -continuous. Furthermore, if X is precompact, then $({}^*({}^V f)(x), f(x)) \in {}^*V$ holds for all $x \in {}^*X$.*

Proof. Fix a symmetric entourage $\sqrt[5]{V}$ of Y such that $\sqrt[5]{V}^5 \subseteq V$. Since $f : {}^*(X, A) \rightarrow (\text{pns}(Y), \text{pns}(B))$, one can define a map ${}^V f : (X, A) \rightarrow (Y, B)$ satisfying $({}^V f(x), f(x)) \in {}^*\sqrt[5]{V}$ on X . Let $x \in X$. By Lemma 2.5, there is an entourage U of X such that $f({}^*U[x]) \subseteq {}^*\sqrt[5]{V}[f(x)]$. For any $y \in U[x]$, $({}^V f(x), f(x)) \in {}^*\sqrt[5]{V}$, $(f(x), f(y)) \in {}^*\sqrt[5]{V}$, $(f(y), {}^V f(y)) \in {}^*\sqrt[5]{V}$, and by the choice of $\sqrt[5]{V}$, $({}^V f(x), {}^V f(y)) \in {}^*V$. By transfer, we have $({}^V f(x), {}^V f(y)) \in V$.

Suppose f is S-continuous on *X . According to Lemma 2.6, U can be chosen independently of $x \in X$. Hence ${}^V f$ is uniformly V -continuous. Suppose in addition that X is precompact. Let $x \in {}^*X$. By [13, Theorem 8.4.34], x is pre-nearstandard. There exists a $\xi \in X$ with $(\xi, x) \in {}^*U$. As we proved above, (by transfer) $({}^*({}^V f)(x), {}^V f(\xi)) \in {}^*\sqrt[5]{V}^3$. Since $({}^V f(\xi), f(\xi)) \in {}^*\sqrt[5]{V}$ and $(f(\xi), f(x)) \in {}^*\sqrt[5]{V}$, we have $({}^*({}^V f)(x), f(x)) \in {}^*V$. \square

2.2. S-homotopy and uniform Vietoris homotopy.

Definition 2.12. Let (X, A) and (Y, B) be uniform pairs. We say that S-continuous maps $f, g : {}^*(X, A) \rightarrow {}^*(Y, B)$ are *S-homotopic* if there is an S-continuous map $h : {}^*(X, A) \times {}^*I \rightarrow {}^*(Y, B)$, called an *S-homotopy* between f and g , such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$.

Proposition 2.13. *S-homotopy is a congruence on \mathbf{UPair}_S .*

Proof. Similar to the ordinary homotopy. \square

This permits us to define the S-homotopy category \mathbf{C}_{Sh} for each full subcategory \mathbf{C} of \mathbf{UPair} . Surprisingly, the full subcategories of \mathbf{TPair}_h , \mathbf{UPair}_h and \mathbf{UPair}_{Sh} whose objects are compact uniform(isable) pairs are isomorphic, and S-continuity and S-homotopy do not provide us anything new.

Theorem 2.14. *Suppose (X, A) and (Y, B) are compact uniform pairs. For any (uniformly) continuous maps $f, g : (X, A) \rightarrow (Y, B)$, the following are equivalent:*

- (1) f and g are homotopic;
- (2) f and g are uniformly homotopic;
- (3) $*f$ and $*g$ are S -homotopic.

Proof. (1) \Rightarrow (2): Let $h : (X, A) \times I \rightarrow (Y, B)$ be a (continuous) homotopy between f and g . By Tychonoff's theorem, $(X, A) \times I$ is compact. h is therefore a uniform homotopy.

(2) \Rightarrow (3): Let $h : (X, A) \times I \rightarrow (Y, B)$ be a uniform homotopy between f and g . By Lemma 2.6, $*h : *(X, A) \times *I \rightarrow *(Y, B)$ is S -continuous. $*h$ is an S -homotopy between $*f$ and $*g$.

(3) \Rightarrow (1): Let $h : *(X, A) \times *I \rightarrow *(Y, B)$ be an S -homotopy between $*f$ and $*g$. By [12, Theorem 4.1.13], we have that $h(*(X \times I)) \subseteq \text{ns}(Y)$ and $h(*(A \times I)) \subseteq \text{ns}(B)$. By Shadow Lemma, there exists a shadow ${}^\circ h : (X, A) \times I \rightarrow (Y, B)$ such that ${}^\circ h(\cdot, 0) = f$ and ${}^\circ h(\cdot, 1) = g$. This is a (uniform) homotopy between f and g . \square

Only two implications (2) \Rightarrow (1) and (2) \Rightarrow (3) can be proved without using compactness. The other implications do not hold in general.

Example 2.15. Let $f : \{\pm 1\} \hookrightarrow \mathbb{R} \setminus \{0\}$ be the inclusion and let $g : \{\pm 1\} \rightarrow \{+1\} \subseteq \mathbb{R} \setminus \{0\}$ be the constant. $*f, *g$ are S -homotopic, but f, g are not homotopic. An example of an S -homotopy between $*f, *g$ is the function $h : \{\pm 1\} \times *I \rightarrow *\mathbb{R} \setminus \{0\}$ defined as follows:

$$h(+1, t) = +1, \quad h(-1, t) = \begin{cases} -1 + 2t, & 0 \leq t < \frac{1}{2}, \\ \varepsilon, & t = \frac{1}{2}, \\ 2t - 1, & \frac{1}{2} < t \leq 1, \end{cases}$$

where ε is a positive infinitesimal.

Example 2.16. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the identity function and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the zero function. f and g are homotopic, but $*f$ and $*g$ are not S -homotopic. To see this, suppose that $*f$ and $*g$ are S -homotopic. Let $h : *\mathbb{R} \times *I \rightarrow *\mathbb{R}$ be an S -homotopy between $*f$ and $*g$. Let n be an infinite hypernatural number. Then,

$$\begin{aligned} n &= h(n, 1) - h(n, 0) \\ &= \sum_{i=0}^{n-1} \left(h\left(n, \frac{i+1}{n}\right) - h\left(n, \frac{i}{n}\right) \right) \\ &\leq \sum_{i=0}^{n-1} \left| h\left(n, \frac{i+1}{n}\right) - h\left(n, \frac{i}{n}\right) \right| \\ &\leq n \cdot \max_i \left| h\left(n, \frac{i+1}{n}\right) - h\left(n, \frac{i}{n}\right) \right|. \end{aligned}$$

Since h is S -continuous, the right hand side is less than n . It is a contradiction.

Definition 2.17. Let $T \geq 0$ be a hyperreal number. Let (X, A) and (Y, B) be uniform pairs. S -continuous maps $f, g : *(X, A) \rightarrow *(Y, B)$ are said to be S_T -homotopic if there exists an S -continuous map $h : *(X, A) \times *[0, T] \rightarrow *(Y, B)$, called an S_T -homotopy between f and g , such that $h(\cdot, 0) = f$ and $h(\cdot, T) = g$.

Proposition 2.18. For every $T \geq 0$, S_T -homotopy is a congruence on \mathbf{UPair}_S .

Proof. We will only prove that S_T -homotopy is compatible with composition. Suppose that S -continuous maps $f, f' : * (X, A) \rightarrow * (Y, B)$ are S_T -homotopic, and so are S -continuous maps $g, g' : * (Y, B) \rightarrow * (Z, C)$. Let $h : * (X, A) \times * [0, T] \rightarrow * (Y, B)$ and $k : * (Y, B) \times * [0, T] \rightarrow * (Z, C)$ be an S_T -homotopy between f and f' and between g and g' , respectively. The map $k \bullet h : * (X, A) \times * [0, T] \rightarrow * (Z, C)$, $(k \bullet h)(x, t) = k(h(x, t), t)$, is an S_T -homotopy between gf and $g'f'$. \square

Proposition 2.19. *If T/T' is finite, S_T -homotopicity implies $S_{T'}$ -homotopicity.*

Proof. Suppose that S -continuous maps $f, g : * (X, A) \rightarrow * (Y, B)$ are S_T -homotopic. Let $h : * (X, A) \times * [0, T] \rightarrow * (Y, B)$ be an S_T -homotopy between f and g . Define an internal map $h' : * (X, A) \times * [0, T'] \rightarrow * (Y, B)$ by $h'(x, t) = h(x, (T/T')t)$. Since T/T' is finite, h' is S -continuous. h' is an $S_{T'}$ -homotopy between f and g . \square

Theorem 2.20. *Suppose $T > 0$ is infinitesimal. For any uniformly continuous maps $f, g : (X, A) \rightarrow (Y, B)$, the following are equivalent:*

- (1) *f and g are pointwise topologically indistinguishable;*
- (2) *$*f$ and $*g$ are S_T -homotopic.*

Proof. (1) \Rightarrow (2): Let $x \in *X$. By transfer, $*f(x)$ and $*g(x)$ are internally topologically indistinguishable, namely, $(*f(x), *g(x)) \in U$ holds for all internal entourage U of $*X$. In particular, for every entourage U of X we have $(*f(x), *g(x)) \in *U$. Hence $*f(x) \approx *g(x)$. The map $h : * (X, A) \times * [0, T] \rightarrow * (Y, B)$ defined by

$$h(x, t) = \begin{cases} *f(x), & 0 \leq t < T, \\ *g(x), & t = T, \end{cases}$$

is an S_T -homotopy between $*f$ and $*g$.

- (2) \Rightarrow (1): Let $h : * (X, A) \times * [0, T] \rightarrow * (Y, B)$ be an S_T -homotopy between $*f$ and $*g$. Let $x \in X$. Since h is S -continuous and $T \approx 0$, $f(x) = h(x, 0) \approx h(x, T) = g(x)$. Hence $f(x)$ and $g(x)$ are topologically indistinguishable. \square

Definition 2.21. S -continuous maps $f, g : * (X, A) \rightarrow * (Y, B)$ are said to be S_\bullet -homotopic provided that they are S_T -homotopic for some $T \geq 0$.

For each full subcategory \mathbf{C} of \mathbf{UPair} , the S_\bullet -homotopy category \mathbf{C}_{S_\bullet} can be defined by replacing “ S -homotopy” with “ S_\bullet -homotopy” in the definition of the S -homotopy category \mathbf{C}_{S_h} .

Theorem 2.22. *Let G be an internal abelian group. μ -homology with coefficients in G satisfies the S_\bullet -homotopy axiom.*

Proof. Similar to [5, Theorem 11]. \square

Example 2.23. The unit open interval $X = (0, 1)$ and the unit circle without one-point $Y = \mathbb{S}^1 \setminus \{(1, 0)\}$ are homotopy equivalent, since they are homeomorphic. However, they are not S_\bullet -homotopy equivalent. To prove this, according to Theorem 2.22, it suffices to show that they have different μ -homology. Since X is uniformly contractible, $H_1^\mu(X; G) = 0$. Y is dense in \mathbb{S}^1 . Using [5, Corollary 13], we have $H_1^\mu(Y; G) = H_1^\mu(\mathbb{S}^1; G) = G$.

Definition 2.24. V -continuous maps $f, g : (X, A) \rightarrow (Y, B)$ are said to be *uniformly V -homotopic*, where V is an entourage of Y , if there is a sequence $\{h_i : (X, A) \rightarrow (Y, B)\}_{i=0}^n$ of uniformly V -continuous maps, called a *uniform V -homotopy* between f and g , such that $(h_i(x), h_{i+1}(x)) \in V$ for all $x \in X$ and $0 \leq i < n$, $h_0 = f$ and $h_n = g$. Uniformly continuous maps $f, g : (X, A) \rightarrow (Y, B)$ are said to be *uniformly Vietoris homotopic* provided that they are uniformly V -homotopic for all entourage V of Y .

Proposition 2.25. *Uniform Vietoris homotopy is a congruence on UPair.*

Proof. Only nontrivial part is the compatibility with composition. Suppose that uniformly continuous maps $f, f' : (X, A) \rightarrow (Y, B)$ are uniformly Vietoris homotopic, and so are uniformly continuous maps $g, g' : (Y, B) \rightarrow (Z, C)$. Let V be any entourage of Z . Let $\{k_i\}_{i=0}^n$ be a uniform V -homotopy between g and g' . There is an entourage W of Y such that $k_i(W[y]) \subseteq V[k_i(y)]$ for all $y \in Y$ and $0 \leq i \leq n$. Let $\{h_j\}_{j=0}^m$ be a uniform W -homotopy between f and f' . Let us define a sequence $\{(k \bullet h)_i : (X, A) \rightarrow (Z, C)\}_{i=0}^{m+n}$ by letting

$$(k \bullet h)_i = \begin{cases} k_0 h_i, & 0 \leq i \leq m, \\ k_{i-m} h_m, & m+1 \leq i \leq m+n. \end{cases}$$

This is a uniform V -homotopy between gf and $g'f'$. The boundary conditions are clear. There is an entourage U of X such that $h_j(U[x]) \subseteq W[h_j(x)]$ for all $x \in X$ and $0 \leq j \leq n$. Then, $k_i h_j(U[x]) \subseteq k_i(W[h_j(x)]) \subseteq V[k_i h_j(x)]$ for all $x \in X$, $0 \leq i \leq n$ and $0 \leq j \leq m$. All $k_i h_j$ s are uniformly V -continuous. In particular, $(k \bullet h)_i$ s are uniformly V -continuous. The values of any two adjacent components are related by V : for $0 \leq i < m$, since $h_{i+1}(x) \in W[h_i(x)]$, we have $(k \bullet h)_{i+1}(x) = k_0 h_{i+1}(x) \in V[k_0 h_i(x)] = V[(k \bullet h)_i(x)]$. For $m \leq i < m+n$, $(k \bullet h)_{i+1}(x) = k_{i+1-m} h_m(x) \in V[k_{i-m} h_m(x)] = V[(k \bullet h)_i(x)]$. \square

Theorem 2.26. *Let $f, g : (X, A) \rightarrow (Y, B)$ be uniformly continuous maps. If $*f$ and $*g$ are S_\bullet -homotopic, then f and g are uniformly Vietoris homotopic.*

Proof. Let $h : *(X, A) \times *[0, T] \rightarrow *(Y, B)$ be an S_T -homotopy between $*f$ and $*g$. Fix an $n \in *\mathbb{N}$ such that T/n is infinitesimal. For $0 \leq i \leq n$ define $h_i = h(\cdot, Ti/n)$. Clearly $h_0 = *f$ and $h_n = *g$. Let V be an entourage of Y . Since h is S -continuous, by Lemma 2.6, all h_i s are internally uniformly $*V$ -continuous. Also $(h_i(x), h_{i+1}(x)) \in *V$ holds for all $x \in *X$ and $0 \leq i < n$. The sequence $\{h_i\}_{i=0}^n$ is an internal uniform $*V$ -homotopy between $*f$ and $*g$. By transfer, f and g are uniformly V -homotopic. \square

Lemma 2.27. *Let G be an abelian group. Let (X, A) and (Y, B) be uniform pairs. Let V be an entourage of Y . Each uniformly V -continuous map $f : (X, A) \rightarrow (Y, B)$ canonically induces a homomorphism $f_V : \check{H}_\bullet^u(X, A; G) \rightarrow H_\bullet(V(Y, Y, V), V(B, Y, V \cap (B \times Y)); G)$. If two uniformly continuous maps $f, g : (X, A) \rightarrow (Y, B)$ are uniformly Vietoris homotopic, then $f_V = g_V$.*

Proof. We simply write X_U for $V(X, X, U)$ and A_U for $V(A, X, U \cap (A \times X))$. The uniform Vietoris homology can be defined as the inverse limit

$$\check{H}_\bullet^u(X, A; G) = \lim_U H_\bullet(X_U, A_U; G),$$

where U runs over all entourages of X . Let $\pi_U : \check{H}_\bullet^u(X, A; G) \rightarrow H_\bullet(X_U, A_U; G)$ be the projection for an entourage U of X . For sufficiently small entourage U of

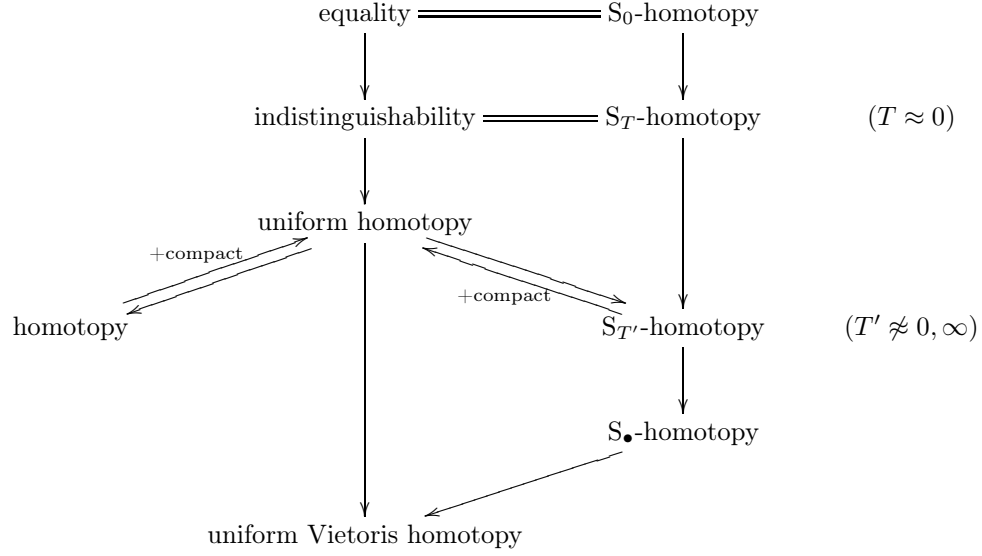


FIGURE 2.1.

X , f is a simplicial map from (X_U, A_U) to (Y_V, B_V) . The induced homomorphism is $f_V = H_\bullet(f; G) \circ \pi_U : \check{H}_\bullet^u(X, A; G) \rightarrow H_\bullet(Y_V, B_V; G)$. This is independent of the choice of U .

Fix an entourage \sqrt{V} of Y with $\sqrt{V}^2 \subseteq V$. Let $\{h_i\}_{i=0}^n$ be a uniform \sqrt{V} -homotopy between f and g . Choose a sufficiently small entourage U of X such that $h_i(U[x]) \subseteq \sqrt{V}[h_i(x)]$ for all $x \in X$ and $0 \leq i \leq n$. All h_i s are simplicial maps from (X_U, A_U) to (Y_V, B_V) . Let us prove that h_i and h_{i+1} are contiguous for all $0 \leq i < n$. Let (a_0, \dots, a_p) be a p -simplex of X_U (or A_U). There is an $x \in X$ with $(a_k, x) \in U$ for all $0 \leq k \leq p$. Since $(h_i(a_k), h_i(x)) \in \sqrt{V}$ and $(h_i(x), h_{i+1}(x)) \in \sqrt{V}$, we have $(h_i(a_k), h_{i+1}(x)) \in V$. Also $(h_{i+1}(a_k), h_{i+1}(x)) \in V$ holds. Hence $(h_i(a_0), \dots, h_i(a_p), h_{i+1}(a_0), \dots, h_{i+1}(a_p))$ is a $(2p+1)$ -simplex of Y_V (or B_V). It follows that $H_\bullet(f; G) = H_\bullet(g; G)$ and $f_V = g_V$. \square

Theorem 2.28. *Let G be an abelian group. Uniform Vietoris homology with coefficients in G satisfies the uniform Vietoris homotopy axiom.*

Proof. Let $f, g : (X, A) \rightarrow (Y, B)$ be uniformly continuous maps. Suppose that they are uniformly Vietoris homotopic. By Lemma 2.27, for any entourage V of Y , $f_V = g_V$ holds. By the definition of the morphism part of \check{H}_\bullet , we have $\check{H}_\bullet^u(f; G) = \check{H}_\bullet^u(g; G)$. \square

Figure 2.1 on page 9 illustrates the relation between standard and nonstandard homotopy equivalences. Arrows indicate implications and double-lines indicate bi-implications. The leftmost node and the center vertical line are in the standard world. The right vertical line is in the nonstandard world.

Question 2.29. *Are there beautiful standard characterisations of S -homotopy and S_\bullet -homotopy?*

2.3. S-connectedness, chain-connectedness and boundedness.

Definition 2.30. A uniform space X is said to be *S-connected* if for any $x, y \in {}^*X$ there is an S-continuous map $\gamma : {}^*I \rightarrow {}^*X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

S-connectedness and path-connectedness are equivalent in compact uniform spaces but not in general.

Theorem 2.31. *Suppose X is a compact uniform space. The following are equivalent:*

- (1) X is path-connected;
- (2) X is S-connected.

Proof. (1) \Rightarrow (2): Let $x, y \in {}^*X$. Since X is compact, x and y are nearstandard. Let ξ and η be points of X infinitely close to x and y , respectively. Let $\gamma : I \rightarrow X$ be a continuous path connecting ξ and η . γ is uniformly continuous, so ${}^*\gamma$ is S-continuous. Define $\delta : {}^*I \rightarrow {}^*X$ by

$$\delta(t) = \begin{cases} x, & 0 \leq t < \frac{1}{3}, \\ {}^*\gamma(3t-1), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ y, & \frac{2}{3} < t \leq 1. \end{cases}$$

This is an S-continuous path connecting x and y . Hence X is S-connected.

(2) \Rightarrow (1): Let $x, y \in X$. Let $\gamma : {}^*I \rightarrow {}^*X$ be an S-continuous path connecting x and y . Since X is compact, by [12, Theorem 4.1.13], we have $\gamma({}^*I) \subseteq \text{ns}(X)$. By Shadow Lemma, there exists a shadow ${}^\circ\gamma : I \rightarrow X$ such that ${}^\circ\gamma(0) = x$ and ${}^\circ\gamma(1) = y$. X is path-connected. □

Example 2.32. \mathbb{R} is path-connected but not S-connected. Suppose contrary that \mathbb{R} is S-connected. Let n be an infinite hypernatural number. Let $\gamma : {}^*I \rightarrow {}^*\mathbb{R}$ be an S-continuous path connecting $-n$ and n . By transfer,

$$\begin{aligned} 2n &= \gamma(1) - \gamma(0) \\ &= \sum_{i=0}^{n-1} \left(\gamma\left(\frac{i+1}{n}\right) - \gamma\left(\frac{i}{n}\right) \right) \\ &\leq \sum_{i=0}^{n-1} \left| \gamma\left(\frac{i+1}{n}\right) - \gamma\left(\frac{i}{n}\right) \right| \\ &\leq n \cdot \max_i \left| \gamma\left(\frac{i+1}{n}\right) - \gamma\left(\frac{i}{n}\right) \right|. \end{aligned}$$

Since γ is S-continuous, the right hand side does not exceed n . It is a contradiction.

Example 2.33. $I \cap \mathbb{Q}$ is S-connected but neither path-connected nor connected. Let $\varepsilon > 0$ be an infinitesimal. $I \cap \mathbb{Q}$ is dense in I . By transfer, for each $x \in {}^*I$, ${}^*(x - \varepsilon, x + \varepsilon) \cap {}^*(I \cap \mathbb{Q})$ is nonempty. There is an internal map $\gamma : {}^*I \rightarrow {}^*(I \cap \mathbb{Q})$ such that $\gamma(x) \in {}^*(x - \varepsilon, x + \varepsilon)$ for all $x \in {}^*I$ and $\gamma(x) = x$ for all $x \in {}^*(I \cap \mathbb{Q})$. For any $x, y \in {}^*(I \cap \mathbb{Q})$, if $x \approx y$, then we have $\gamma(x) \approx x \approx y \approx \gamma(y)$. γ is S-continuous. For any $x, y \in {}^*(I \cap \mathbb{Q})$, the S-continuous path $\gamma_{x,y}(t) = \gamma((1-t)x + ty)$ connects x and y .

Definition 2.34. Let $T \geq 0$ be a hyperreal number. A uniform space X is said to be S_T -connected if for any $x, y \in {}^*X$ there is an S-continuous map $\gamma : {}^*[0, T] \rightarrow {}^*X$ with $\gamma(0) = x$ and $\gamma(T) = y$.

Proposition 2.35. If T/T' is finite, S_T -connectedness implies $S_{T'}$ -connectedness.

Proof. Suppose a uniform space X is S_T -connected. For any $x, y \in {}^*X$, let $\gamma_{x,y} : {}^*[0, T] \rightarrow {}^*X$ be an S-continuous path connecting x and y . Define $\gamma'_{x,y} : {}^*[0, T'] \rightarrow {}^*X$ by $\gamma'_{x,y}(t) = \gamma_{x,y}((T/T')t)$. Since T/T' is finite, $\gamma'_{x,y}$ is S-continuous. X is $S_{T'}$ -connected. \square

Theorem 2.36. Suppose $T > 0$ is infinitesimal. Let X be a uniform space. The following are equivalent:

- (1) X is indiscrete;
- (2) X is S_T -connected.

Proof. (1) \Rightarrow (2): All points of X are topologically indistinguishable. By transfer, for any $x, y \in {}^*X$, we have $x \approx y$. The map

$$\gamma_{x,y}(t) = \begin{cases} x, & t < T, \\ y, & t = T \end{cases}$$

is an S-continuous path connecting x and y . X is S_T -connected.

- (2) \Rightarrow (1): For any $x, y \in X$, there is an S-continuous path $\gamma_{x,y} : {}^*[0, T] \rightarrow {}^*X$ connecting x and y . Since $T \approx 0$, $x = \gamma_{x,y}(0) \approx \gamma_{x,y}(T) = y$. x, y are topologically indistinguishable. X is indiscrete. \square

Definition 2.37. A uniform space X is said to be S_\bullet -connected if X is S_T -connected for some $T \geq 0$. X is said to be S_∞ -connected if for any $x, y \in {}^*X$ there are a hyperreal number $T \geq 0$ and an S-continuous map $\gamma : {}^*[0, T] \rightarrow {}^*X$ with $\gamma(0) = x$ and $\gamma(T) = y$.

Recall that a uniform space X is said to be *chain-connected* if for each entourage U of X any two points of X can be connected by a U -chain, and a subset B of X is said to be *bounded* (in the sense of Hu [4]) if for each entourage U of X there is an $n \in \mathbb{N}$ such that any two points of B can be connected by a U -chain of length n in X .

Theorem 2.38. Let X be a uniform space. The following are equivalent:

- (1) X is bounded in itself;
- (2) X is S_\bullet -connected.

Proof. (1) \Rightarrow (2): By saturation, there is an internal symmetric entourage U of *X with $U \subseteq (\approx)$. By transfer, there exists an $n \in {}^*\mathbb{N}$ such that for any $x, y \in {}^*X$ there exists an internal U -chain of length n connecting x and y . Let $x, y \in {}^*X$ and $\{x_i\}_{i=0}^{n-1}$ be an internal U -chain of length n connecting x and y . Let us define an internal map $\gamma : {}^*[0, n-1] \rightarrow {}^*X$ by letting

$$\gamma(t) = \begin{cases} x_i, & i \leq t < i+1 \\ x_{n-1}, & t = n-1. \end{cases}$$

Since $(\gamma(t), \gamma(u)) \in U$ for all $t \approx u$, γ is an S-continuous path connecting x and y . X is S_{n-1} -connected.

(2) \Rightarrow (1): Suppose X is S_T -connected for some $T \geq 0$. Let U be an entourage of X . Fix an $n \in {}^*\mathbb{N}$ such that T/n is infinitesimal. Let $x, y \in {}^*X$. There exists an S -continuous path $\gamma : {}^*[0, T] \rightarrow {}^*X$ connecting x and y . The sequence $\{\gamma(Ti/n)\}_{i=0}^n$ is an internal *U -chain of length $n+1$ connecting x and y . By transfer, there exists an $n \in \mathbb{N}$ such that for any $x, y \in X$ there exists a U -chain of length $n+1$ connecting x and y . \square

Definition 2.39. Let $T \geq 0$ be a hyperreal number. Let X be a uniform space. A subset B of X is said to be S_T -bounded if for any $x, y \in {}^*B$ there is an S -continuous map $\gamma : {}^*[0, T] \rightarrow {}^*X$ with $\gamma(0) = x$ and $\gamma(T) = y$, and B is said to be S_\bullet -bounded if B is S_T -bounded for some $T \geq 0$.

Theorem 2.40. Let X be a uniform space and B a subset of X . The following are equivalent:

- (1) B is bounded in X ;
- (2) B is S_\bullet -bounded in X .

Proof. Similar to Theorem 2.38. \square

Theorem 2.41. Let X be a uniform space. The following are equivalent:

- (1) X is chain-connected;
- (2) X is S_∞ -connected.

Proof. (1) \Rightarrow (2): Let $x, y \in {}^*X$. By [5, Lemma 7], there exists an infinitesimal chain $\{x_i\}_{i=0}^n$ connecting x and y . Let us define an internal map $\gamma : {}^*[0, n-1] \rightarrow {}^*X$ by letting

$$\gamma(t) = \begin{cases} x_i, & i \leq t < i+1 \\ x_{n-1}, & t = n-1. \end{cases}$$

Clearly γ is an S -continuous path connecting x and y . X is S_∞ -connected.

(2) \Rightarrow (1): Let $x, y \in {}^*X$. There exists an S -continuous path $\gamma : {}^*[0, T] \rightarrow {}^*X$ connecting x and y . Fix an $n \in {}^*\mathbb{N}$ such that T/n is infinitesimal. The sequence $\{\gamma(Ti/n)\}_{i=0}^n$ is an infinitesimal chain connecting x and y . By [5, Lemma 7], X is chain-connected. \square

Corollary 2.42. Suppose X is a precompact uniform space. The following are equivalent:

- (1) X is S_\bullet -connected;
- (2) X is S_∞ -connected.

Proof. We only need to show that every precompact chain-connected space is bounded in itself, but this is a special case of [4, Theorem V]. \square

Corollary 2.43. For any S -continuous maps $f, g : {}^*(X, A) \rightarrow {}^*(Y, B)$, the following are equivalent:

- (1) f, g are S_\bullet -homotopic;
- (2) there are a chain-connected uniform space Z , an S -continuous map $h : {}^*(X, A) \times {}^*Z \rightarrow {}^*(Y, B)$ and $z_0, z_1 \in {}^*Z$ such that $h(\cdot, z_0) = f$ and $h(\cdot, z_1) = g$.

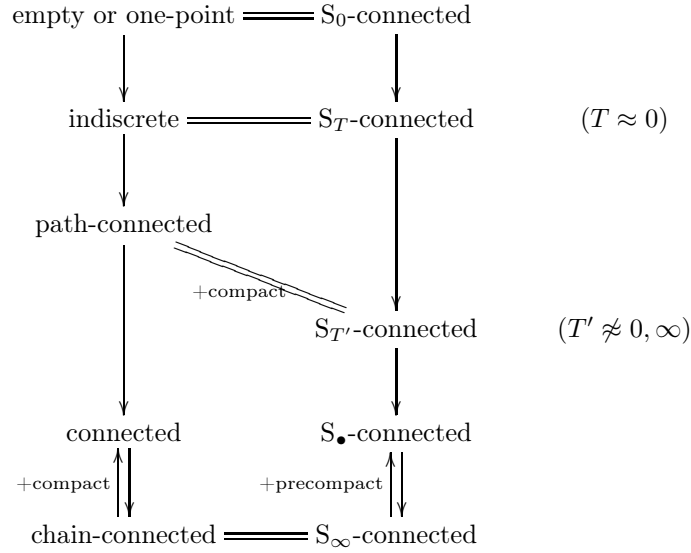


FIGURE 2.2.

Figure 2.2 on page 13 illustrates the relation between standard and nonstandard connectedness. The left vertical line is in the standard world. The right vertical line is in the nonstandard world.

Question 2.44. *Is there a beautiful standard characterisation of S -connectedness?*

3. EQUIVALENCE BETWEEN UNIFORM VIETORIS HOMOLOGY AND μ -HOMOLOGY

3.1. Compact case.

Lemma 3.1 (Lebesgue Covering Lemma). *Let X be a compact uniform space. Every open cover of X is a uniform cover of X .*

Proof. Let \mathcal{U} be an open cover of X . By saturation, there exists an internal entourage V of *X with $V \subseteq (\approx)$. One may assume that $\mathcal{V} = \{{}^*V[x] \mid x \in {}^*X\}$ is an internal open cover of *X . Suppose $x \in {}^*X$. By [12, Theorem 4.1.13], x is nearstandard. Let ξ be a point of X infinitely close to x . There exists an open neighbourhood $U \in \mathcal{U}$ of ξ . We have that $V[x] \subseteq \mu(x) = \mu(\xi) \subseteq {}^*U$. \mathcal{V} is an internal refinement of ${}^*\mathcal{U}$, and therefore ${}^*\mathcal{U}$ is an internal uniform cover of *X . By transfer, \mathcal{U} is a uniform cover of X . \square

Corollary 3.2. *Let G be an abelian group. For compact uniform pairs, uniform Vietoris homology with coefficients in G coincides with Vietoris homology with the same coefficients.*

Proof. Immediately from Lebesgue Covering Lemma. \square

Theorem 3.3. *Let G be an abelian group. Uniform Vietoris homology with coefficients in *G coincides with μ -homology with the same coefficients for all compact uniform pairs.*

Proof. In the compact case, μ -homology agrees with McCord homology (see [5, Section 6]). McCord homology is isomorphic to Čech homology ([7, Theorem 9]) and hence to Vietoris homology ([2, Theorem 2a]). By Corollary 3.2, they coincide with uniform Vietoris homology. \square

In the noncompact case, uniform Vietoris homology may not be isomorphic to Vietoris homology. An easy example is \mathbb{Q}/\mathbb{Z} . Since \mathbb{Q}/\mathbb{Z} has disjoint open covers as fine as one likes, the 1-st Vietoris homology of \mathbb{Q}/\mathbb{Z} vanishes. On the other hand, the 1-st uniform Vietoris homology of \mathbb{Q}/\mathbb{Z} does not vanish (except for the trivial case $G = 0$). Furthermore, \mathbb{Q}/\mathbb{Z} and \mathbb{R}/\mathbb{Z} have the same uniform Vietoris homology. This can be generalised as stated later.

3.2. Precompact case.

Theorem 3.4. *Let G be an abelian group. The domain category of $\check{H}_\bullet^u(\cdot; G)$ can be replaced with \mathbf{pUPair}_S .*

Proof. We shall prove that every S -continuous map between precompact uniform pairs canonically induces a homomorphism. Let $\pi_U : \check{H}_\bullet^u(X, A; G) \rightarrow H_\bullet(X_U, A_U; G)$ denote the projection for an entourage U of X . For each entourage V , fix an entourage \sqrt{V} with $\sqrt{V}^2 \subseteq V$.

Suppose that $f : * (X, A) \rightarrow * (Y, B)$ is an S -continuous map, where (X, A) and (Y, B) are precompact uniform pairs. By [13, Theorem 8.4.34], we have that $f(*X) \subseteq \text{pns}(Y)$ and $f(*A) \subseteq \text{pns}(B)$. By Preshadow Lemma, for each entourage V of Y , there exists a \sqrt{V} -preshadow ${}^V f : (X, A) \rightarrow (Y, B)$. For sufficiently small entourage U of X , ${}^V f$ is a simplicial map from (X_U, A_U) to (Y_V, B_V) . The induced homomorphism is $f_V = H_\bullet({}^V f; G) \circ \pi_U : \check{H}_\bullet^u(X, A; G) \rightarrow H_\bullet(Y_V, B_V; G)$.

On the other hand, by Lemma 2.6, there exists an entourage U of X such that $f(*U[x]) \subseteq *\sqrt{V}[f(x)]$ for all $x \in *X$. For simplicity, we write ${}^V f = *({}^V f)$. We may assume that ${}^V f(*U[x]) \subseteq *\sqrt{V}[{}^V f(x)]$ also holds for all $x \in *X$. Let us prove that f and ${}^V f$ are internally contiguous as internal simplicial maps from $* (X_U, A_U)$ to $* (Y_V, B_V)$. Let (a_0, \dots, a_p) be an internal p -simplex of $*X_U$ (or $*A_U$). There is an $x \in *X$ with $(a_k, x) \in *U$ for all $0 \leq k \leq p$. Since $(f(a_k), f(x)) \in *\sqrt{V}$ and $(f(x), {}^V f(x)) \in *\sqrt{V}$, we have $(f(a_k), {}^V f(x)) \in *V$. Also $({}^V f(a_k), {}^V f(x)) \in *V$ holds. Hence $(f(a_0), \dots, f(a_p), {}^V f(a_0), \dots, {}^V f(a_p))$ is an internal $(2p+1)$ -simplex of $*Y_V$ (or $*B_V$).

Now, we will prove that the following diagram commutes for all $V \supseteq W$:

$$\begin{array}{ccc} & \check{H}_\bullet^u(X, A; G) & \\ f_V \swarrow & & \searrow f_W \\ H_\bullet(Y_V, B_V; G) & \xleftarrow{p_{WV}} & H_\bullet(Y_W, B_W; G) \end{array}$$

As we proved above, f and ${}^W f$ are internally contiguous, and so are f and ${}^V f$. It follows that $*(H_\bullet({}^V f; G)) = (*H_\bullet)(f; *G) = *(p_{WV} \circ H_\bullet({}^W f; G))$. By transfer, we have that $H_\bullet({}^V f; G) = p_{WV} \circ H_\bullet({}^W f; G)$ and $f_V = p_{WV} \circ f_W$. By the universal property of $\check{H}_\bullet^u(Y, B; G)$, we have the homomorphism $\check{H}_\bullet^u(f; G) : \check{H}_\bullet^u(X, A; G) \rightarrow \check{H}_\bullet^u(Y, B; G)$. We can also show that $\check{H}_\bullet^u(f; G)$ is independent of the choice of preshadows.

Suppose that $f : {}^*(X, A) \rightarrow {}^*(Y, B)$ and $g : {}^*(Y, B) \rightarrow {}^*(Z, C)$ are S-continuous maps, where (X, A) , (Y, B) and (Z, C) are precompact uniform pairs. Let V be an entourage of Z . Let Vg be a \sqrt{V} -preshadow of g . There is an entourage U of Y such that ${}^Vg(U[x]) \subseteq \sqrt{V}[{}^Vg(x)]$. Let Uf be a \sqrt{U} -preshadow of f . Then, ${}^Vg^Uf$ is a \sqrt{V} -preshadow of gf . There is an entourage W of X such that ${}^Uf(W[x]) \subseteq \sqrt{U}[{}^Uf(x)]$ and ${}^Vg^Uf(W[x]) \subseteq \sqrt{V}[{}^Vg^Uf(x)]$ for all $x \in X$. The following diagram is commutative:

$$\begin{array}{ccccc}
 & & \check{H}_\bullet^u(g; G) \circ \check{H}_\bullet^u(f; G) & & \\
 & \swarrow & & \searrow & \\
 \check{H}_\bullet^u(X, A; G) & \xrightarrow{\check{H}_\bullet^u(f; G)} & \check{H}_\bullet^u(Y, B; G) & \xrightarrow{\check{H}_\bullet^u(g; G)} & \check{H}_\bullet^u(Z, C; G) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_\bullet(X_W, A_W; G) & \xrightarrow{H_\bullet({}^Uf; G)} & H_\bullet(Y_U, B_U; G) & \xrightarrow{H_\bullet({}^Vg; G)} & H_\bullet(Z_V, C_V; G) \\
 & \swarrow & & \searrow & \\
 & & H_\bullet({}^Vg^Uf; G) & &
 \end{array}$$

We have $\check{H}_\bullet^u(gf; G) = \check{H}_\bullet^u(g; G) \circ \check{H}_\bullet^u(f; G)$. \square

Theorem 3.5. *Let G be an abelian group. Uniform Vietoris homology with coefficients in G satisfies the S_\bullet -homotopy axiom for all precompact uniform pairs.*

Proof. Suppose that S-continuous maps $f, g : {}^*(X, A) \rightarrow {}^*(Y, B)$ are S_\bullet -homotopic, where (X, A) and (Y, B) are precompact uniform pairs. Let $h : {}^*(X, A) \times {}^*[0, T] \rightarrow {}^*(Y, B)$ be an S_T -homotopy between f and g . Fix an $n \in {}^*\mathbb{N}$ such that T/n is infinitesimal. Define $h_i = h(\cdot, i/n)$. Let V be an entourage of Y . Let \sqrt{V} be an entourage of Y with $\sqrt{V}^2 \subseteq V$. By Lemma 2.6, there exists an entourage U of X such that $h_i({}^*U[x]) \subseteq {}^*\sqrt{V}[h_i(x)]$ for all $x \in {}^*X$ and $0 \leq i \leq n$. As similar to the proof of Lemma 2.27, we can prove that any two adjacent maps from $\{h_i\}_{i=0}^n$ are internally contiguous as internal simplicial maps from ${}^*(X_U, A_U)$ to ${}^*(Y_V, B_V)$. It follows that $({}^*H_\bullet)(f; {}^*G) = ({}^*H_\bullet)(g; {}^*G)$ and ${}^*f_V = {}^*g_V$. By transfer, we have $f_V = g_V$ and therefore $\check{H}_\bullet^u(f; G) = \check{H}_\bullet^u(g; G)$. \square

Lemma 3.6. *Suppose that (Y, B) is a dense subpair of a uniform pair (X, A) . Then there exists an S-deformation retraction $r : {}^*(X, A) \rightarrow {}^*(Y, B)$.*

Proof. The proof is similar to [5, Theorem 12]. Note that in the construction of an S-deformation retraction $r : {}^*X \rightarrow {}^*Y$ one may assume that $r({}^*A) \subseteq {}^*B$. \square

Corollary 3.7. *Let G be an internal abelian group. Suppose that (Y, B) is a dense subpair of a uniform pair (X, A) . Then the inclusion map $i : (Y, B) \hookrightarrow (X, A)$ induces the isomorphism $H_\bullet^\mu(i; G) : H_\bullet^\mu(Y, B; G) \cong H_\bullet^\mu(X, A; G)$.*

Corollary 3.8. *Let G be an abelian group. Suppose that (Y, B) is a dense subpair of a precompact uniform pair (X, A) . Then the inclusion map $i : (Y, B) \hookrightarrow (X, A)$ induces the isomorphism $\check{H}_\bullet^u(i; G) : \check{H}_\bullet^u(Y, B; G) \cong \check{H}_\bullet^u(X, A; G)$.*

Theorem 3.9. *Let G be an abelian group. μ -homology with coefficients in *G coincides with uniform Vietoris homology with the same coefficients for all precompact uniform pairs.*

Proof. Let (X, A) be a precompact uniform pair. Let \bar{X} be a uniform completion of X . Let \bar{A} be the closure of A in \bar{X} . Since X is precompact, \bar{X} is compact, and so is \bar{A} . By Corollary 3.7, (X, A) and (\bar{X}, \bar{A}) have the same μ -homology. By Theorem 3.3, the μ -homology of (\bar{X}, \bar{A}) is isomorphic to the uniform Vietoris homology of (\bar{X}, \bar{A}) . Finally, by Corollary 3.8, the uniform Vietoris homology of (\bar{X}, \bar{A}) is isomorphic to that of (X, A) . The proof is completed. \square

The equivalence does not hold in general. The reason is that μ -homology does not preserve infinite coproducts except for the trivial case $G = 0$.

Theorem 3.10. *There exist a uniform space X and an abelian group G such that $H^\mu_\bullet(X; *G) \neq \check{H}^u_\bullet(X; *G)$.*

Proof. Let X be the discrete space \mathbb{N} . Let G be any nontrivial finite abelian group. Then, by transfer, $G = *G$. By the additivity of \check{H}^u_\bullet , we have $\check{H}^u_0(X; G) = G^{\oplus \mathbb{N}}$. The cardinality of $\check{H}^u_0(X; G)$ is \aleph_0 . On the other hand, we have $H^\mu_0(X; G) = *(G^{\oplus \mathbb{N}})$. By saturation, every element of $G^{\oplus \mathbb{N}}$ can be extended to an element of $*(G^{\oplus \mathbb{N}})$. The cardinality of $H^\mu_0(X; G)$ is at least 2^{\aleph_0} . Hence $\check{H}^u_0(X; G)$ and $H^\mu_0(X; G)$ are not isomorphic. \square

Question 3.11. *Does the equivalence of μ -homology and uniform Vietoris homology hold for more general uniform spaces? What is the limitation of the equivalence?*

3.3. Uniform Vietoris homology via finite uniform covers. Let (X, A) be a uniform pair. Let $\text{Cov}_{uf}(X)$ denote the set of all finite uniform covers of X . Define

$$\check{H}^{uf}_\bullet(X, A; G) = \lim_{\mathcal{U} \in \text{Cov}_{uf}(X)} H_\bullet(X_{\mathcal{U}}, A_{\mathcal{U}}; G).$$

If X is precompact, we have $\check{H}^{uf}_\bullet(X, A; G) = \check{H}^u_\bullet(X, A; G)$, so $\check{H}^{uf}_\bullet(X, A; *G) = H^\mu_\bullet(X, A; *G)$. The equivalence does not hold in general.

Theorem 3.12. *There exist a uniform space X and an abelian group G such that $H^\mu_\bullet(X; *G) \neq \check{H}^{uf}_\bullet(X; *G)$.*

Proof. Let X and G be the same as in the proof of Theorem 3.10. The cardinality of $\check{H}^{uf}_0(X; G)$ is at most $2^{2^{\aleph_0}}$. Let A be a set of cardinality exceeding $2^{2^{\aleph_0}}$. By saturation, there is an internal hyperfinite set A' with $A \subseteq A'$. Since $G^{\oplus \mathbb{N}}$ is infinite, by transfer, A' can be internally embedded into $*(G^{\oplus \mathbb{N}})$. The cardinality of $*(G^{\oplus \mathbb{N}})$ is at least that of A' . The cardinality of $H^\mu_0(X; G)$ exceeds $2^{2^{\aleph_0}}$. $\check{H}^{uf}_0(X; G)$ and $H^\mu_0(X; G)$ are not isomorphic. \square

3.4. Uniform shape invariance.

Corollary 3.13. *Let G be an abelian group. μ -homology with coefficients in $*G$ is a uniform shape invariant in the sense of Doitchinov [1] for precompact metric spaces.*

Proof. According to [1, Theorem 1], uniform Čech homology is a uniform shape invariant for metric spaces, and so is uniform Vietoris homology. By Theorem 3.9, μ -homology is a uniform shape invariant for precompact metric spaces. \square

Corollary 3.14. *Let G be an abelian group. μ -homology with coefficients in $*G$ is a uniform shape invariant in the sense of Miyata [10] for precompact uniform spaces.*

Proof. Recall that a uniform space X is said to be *finitistic* if every uniform cover of X has a uniform refinement of finite order. Here the order of a cover \mathcal{U} is the smallest $n \in \mathbb{N} \setminus \{-1\}$ such that any $n+2$ distinct members of \mathcal{U} do not intersect. If there is no such an n , the order of \mathcal{U} is infinite. It is obvious that every precompact uniform space is finitistic. [10, Theorem 6.1] states that uniform Čech homology is a uniform shape invariant for finitistic uniform spaces. μ -homology is a uniform shape invariant for precompact uniform spaces. \square

Question 3.15. *Does the uniform shape invariance of μ -homology hold for more general uniform spaces?*

4. KORPPI HOMOLOGY AND EXCISION AND CONTINUITY AXIOMS FOR μ -HOMOLOGY

4.1. Korppi homology. Korppi [8] introduced another McCord-type homology of *completely regular* spaces. Let H_\bullet^K denote the Korppi homology functor. Korppi homology is defined only for coefficient groups of the form *G , but can be defined also for internal coefficient groups. We will prove that the Korppi homology of a completely regular space is isomorphic to the μ -homology of the associated fine uniform space.

The definition of Korppi homology is similar to that of μ -homology. Only difference is the definition of the infinite closeness. Korppi's definition uses the concept of normal covers instead of uniform covers. Let X be a topological space. $\mathcal{V} \subseteq \mathcal{P}(X)$ is called a *star-refinement* of $\mathcal{U} \subseteq \mathcal{P}(X)$ (denoted by $\mathcal{U} \preceq^* \mathcal{V}$) provided that $\mathcal{U} \preceq \{\text{St}(V, \mathcal{V}) \mid V \in \mathcal{V}\}$, where $\text{St}(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$. An open cover \mathcal{U} of X is said to be *normal* if there exists a sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of open covers of X such that $\mathcal{U}_0 = \mathcal{U}$ and $\mathcal{U}_n \preceq^* \mathcal{U}_{n+1}$ for all $n \in \mathbb{N}$. Let $x, y \in {}^*X$. We say that x and y are *infinitely close* (denoted by $x \sim y$) if for every normal cover \mathcal{U} , there exists a $U \in {}^*\mathcal{U}$ such that $x, y \in U$.

Let $F : \mathbf{CRPair} \rightarrow \mathbf{UPair}$ be the left adjoint functor of the forgetful (topologisation) functor $U : \mathbf{UPair} \rightarrow \mathbf{CRPair}$. The underlying set of FX is the same as X . The uniformity of FX is the finest uniformity compatible with the topology of X . A uniform space Y is said to be *fine* if $Y = FUY$.

Lemma 4.1. *Let X be a completely regular space. For any $x, y \in {}^*X$, $x \sim_X y$ if and only if $x \approx_{FX} y$.*

Proof. Suppose that $x \sim_X y$. Let U be an entourage of FX . Fix a symmetric open entourage V of FX with $V \circ V \subseteq U$. By [6, Theorem 20], $\mathcal{V} = \{V[z] \mid z \in X\}$ is a normal cover of X . By transfer, ${}^*\mathcal{V} = \{{}^*V[z] \mid z \in {}^*X\}$. Since $x \sim_X y$, there exists a $z \in {}^*X$ with $x, y \in {}^*V[z]$. We have that $(x, z) \in {}^*V$, $(z, y) \in {}^*V$ so $(x, y) \in {}^*U$. Hence $x \approx_{FX} y$.

Conversely, suppose that $x \approx_{FX} y$. Let \mathcal{U} be a normal cover. By [6, Theorem 20], \mathcal{U} is a uniform cover of FX . There exists an entourage U of FX such that $\mathcal{U} \preceq \{U[z] \mid z \in X\}$. Since $x \approx_{FX} y$, we have that $(x, y) \in {}^*U$ and therefore $y \in {}^*U[x]$. By transfer, there exists a $V \in {}^*\mathcal{U}$ with $x, y \in {}^*U[x] \subseteq V$. Hence $x \sim_X y$. \square

Note that for a topological pair (X, A) , it is possible that $x \sim_X y$ but $x \not\sim_A y$ (see [8, Remark 8]). A is said to be *normally embedded* if in X for every normal cover \mathcal{V} of A , there exists a normal cover \mathcal{U} of X such that $\mathcal{V} \preceq \{U \cap A \mid U \in \mathcal{U}\}$. If A is normally embedded in X , then \sim_A agrees with \sim_X on *A ([8, Lemma 9]).

Theorem 4.2. *Let G be an internal abelian group. For any completely regular space X and any normally embedded subspace A of X , we have that $H_{\bullet}^K(X, A; G) = H_{\bullet}^{\mu}(F(X, A); G) = H_{\bullet}^{\mu}(FX, FA; G)$.*

Theorem 4.3. *Let G be an internal abelian group. For any fine uniform space X and any fine subspace A of X , we have that $H_{\bullet}^{\mu}(X, A; G) = H_{\bullet}^K(U(X, A); G) = H_{\bullet}^K(UX, UA; G)$.*

We can say that μ -homology is a generalisation of Korppi homology from fine uniform spaces to arbitrary uniform spaces. μ -homology inherits many nice properties from Korppi homology (see e.g. Theorem 4.9). Korppi proved that Čech homology with standard coefficients can be naturally embedded into Korppi homology with nonstandard coefficients ([8, Theorem 76]). The analogue holds between uniform Vietoris homology and μ -homology.

Theorem 4.4. *Let G be an abelian group. There exists a natural monomorphism from $\check{H}_{\bullet}^{\mu}(\cdot; G)$ to $H_{\bullet}^{\mu}(\cdot; *G)$.*

Proof. The proof is completely analogous to the proof of [8, Theorem 76]. \square

4.2. Excision axiom for μ -homology. As proved in [5, Proposition 3], μ -homology satisfies the weak excision axiom. We will prove that μ -homology satisfies the excision axiom.

Definition 4.5. Let X be a uniform space. Let A and B be subsets of X . A is said to be *strongly contained in* B (denoted by $A \Subset B$) if $\text{St}(A, \mathcal{U}) \subseteq B$ for some uniform cover \mathcal{U} of X .

Lemma 4.6. *Let X be a uniform space. Let A and B be subsets of X . Then $A \Subset B$ if and only if $\mu(*A) \subseteq *B$, where $\mu(*A) = \{x \in *X \mid *A \cap \mu(x) \neq \emptyset\}$.*

Proof. Suppose that $\mu(*A) \subseteq *B$. By saturation, there exists an internal uniform cover \mathcal{U} such that $*\mathcal{V} \preceq \mathcal{U}$ for all uniform cover \mathcal{V} . Then $\text{St}(*A, *\mathcal{U}) \subseteq \mu(*A) \subseteq *B$. By transfer, we have $A \Subset B$. Conversely, suppose $A \Subset B$. Let \mathcal{U} be a uniform cover with $\text{St}(A, \mathcal{U}) \subseteq B$. Then, by transfer, we have $\mu(*A) \subseteq \text{St}(*A, *\mathcal{U}) \subseteq *B$. \square

Theorem 4.7 (Excision). *Let X be a uniform space. Let A and B be subsets of X . If $X \setminus A \Subset B$ (or $X \setminus B \Subset A$), then the inclusion map $i : (A, A \cap B) \hookrightarrow (X, B)$ induces the isomorphism $H_{\bullet}^{\mu}(i; G) : H_{\bullet}^{\mu}(A, A \cap B; G) \cong H_{\bullet}^{\mu}(X, B; G)$.*

Proof. The proof is similar to [5, Proposition 3]. We shall only prove that any microsimplex σ on X lies in either $*A$ or $*B$. Suppose that σ does not lie in $*A$. Then σ intersects $*(X \setminus A)$. All vertices of σ are in $\mu(*(X \setminus A))$. By Lemma 4.6, σ lies in $*B$. \square

μ -homology satisfies the uniform homotopy, exactness, excision and dimension axioms (see [5, Section 3]). In other words, μ -homology is an *exact* uniform homology theory in the sense of Miyata [11].

4.3. Continuity axiom for μ -homology.

Definition 4.8. Let $\mathbf{X} = (I, X_i, p_{ji})$ be an inverse system of uniform spaces. Let $\pi : X \rightarrow \mathbf{X}$ be a cone over \mathbf{X} . \mathbf{X} is called a *uniform resolution* of X if the following conditions hold:

- (1) $\forall \mathcal{U} \in \text{Cov}_u(X) \exists i \in I \exists \mathcal{V} \in \text{Cov}_u(X_i) \mathcal{U} \preceq \pi_i^{-1}\mathcal{V}$;

$$(2) \quad \forall i \in I \forall \mathcal{V} \in \text{Cov}_u(X_i) \exists j \in I \pi_{ji}(X_j) \subseteq \text{St}(\pi_i(X), \mathcal{V}).$$

Let $(\mathbf{X}, \mathbf{A}) = (I, X_i, A_i, p_{ji})$ be an inverse system of uniform spaces. Let $\pi : (X, A) \rightarrow (\mathbf{X}, \mathbf{A})$ be a cone over (\mathbf{X}, \mathbf{A}) . (\mathbf{X}, \mathbf{A}) is called a uniform resolution of (X, A) if $\mathbf{X} = (I, X_i, p_{ji})$ and $\mathbf{A} = (I, A_i, p_{ji})$ are uniform resolutions of X and A , respectively.

Theorem 4.9 (Continuity). *Let $(\mathbf{X}, \mathbf{A}) = (I, X_i, A_i, p_{ji})$ be an inverse system of uniform pairs. Let $\pi : (X, A) \rightarrow (\mathbf{X}, \mathbf{A})$ be a cone over (\mathbf{X}, \mathbf{A}) . If (\mathbf{X}, \mathbf{A}) is a uniform resolution of (X, A) , then π induces an isomorphism $H_\bullet^\mu(X, A; G) \cong \lim_{i \in I} H_\bullet^\mu(X_i, A_i; G)$.*

The proof is completely analogous to the proof of [8, Theorem 71]. The original theorem can be seen as the special case for fine uniform spaces. For example, in the proof, we shall replace [8, Lemma 60] with the following lemma. Observe that the proof of the modified lemma is completely same as the original one.

Definition 4.10. Let $\mathbf{X} = (I, X_i, p_{ji})$ be an inverse system of uniform spaces. Let $J = \{i \in I \mid \forall j \in I (j \leq i)\}$. For $x, y \in {}^*X_j$ ($j \in J$), we say that $x \approx_{\mathbf{X}} y$ provided that for any $i \in I$ we have ${}^*\pi_{ji}(x) \approx_{X_i} {}^*\pi_{ji}(y)$.

Lemma 4.11. *Let $\mathbf{X} = (I, X_i, p_{ji})$ be an inverse system of uniform spaces. Let $\pi : X \rightarrow \mathbf{X}$ be a cone over \mathbf{X} . The following are equivalent:*

- (1) $\forall \mathcal{U} \in \text{Cov}_u(X) \exists i \in I \exists \mathcal{V} \in \text{Cov}_u(X_i) \mathcal{U} \preceq \pi_i^{-1}\mathcal{V}$;
- (2) for any $x, y \in {}^*X$, $x \approx_X y$ if and only if ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ for all $i \in I$;
- (3) for any $x, y \in {}^*X$, $x \approx_X y$ if and only if ${}^*\pi_j(x) \approx_{\mathbf{X}} {}^*\pi_j(y)$ for all $j \in J$;
- (4) for any $x, y \in {}^*X$, $x \approx_X y$ if and only if ${}^*\pi_j(x) \approx_{\mathbf{X}} {}^*\pi_j(y)$ for some $j \in J$.

Proof. Every finite subset of I has an upper bound in I . By saturation, I has an upper bound in *I . Hence J is nonempty.

- (1) \Rightarrow (2): Suppose that $x \not\approx_X y$. There exists a uniform cover \mathcal{U} of X such that x and y are not ${}^*\mathcal{U}$ -near. There exist an $i \in I$ and a uniform cover \mathcal{V} of X_i with $\mathcal{U} \preceq \pi_i^{-1}\mathcal{V}$. x and y are not ${}^*(\pi_i^{-1}\mathcal{V})$ -near. Therefore ${}^*\pi_i(x)$ and ${}^*\pi_i(y)$ are not ${}^*\mathcal{V}$ -near. Hence ${}^*\pi_i(x) \not\approx_{X_i} {}^*\pi_i(y)$. The converse follows from Lemma 2.6.
- (2) \Rightarrow (1): Suppose, on the contrary, that there exists a uniform cover \mathcal{U} of X such that for any $i \in I$ and any uniform cover \mathcal{V} of X_i we have $\mathcal{U} \not\preceq \pi_i^{-1}\mathcal{V}$. Let \mathcal{U}' be a uniform star-refinement of \mathcal{U} . Let $i_k \in I$ and $\mathcal{V}_k \in \text{Cov}_u(X_{i_k})$ for $1 \leq k \leq n$. Choose an upper bound $j \in I$ of $\{i_k \mid 1 \leq k \leq n\}$ and an upper bound \mathcal{V} of $\{\pi_{i_k j}^{-1}\mathcal{V}_k \mid 1 \leq k \leq n\}$. Since $\mathcal{U} \not\preceq \pi_j^{-1}\mathcal{V}$, there is a $V \in \pi_j^{-1}\mathcal{V}$ such that V is not contained in any member of \mathcal{U} . Clearly $V \neq \emptyset$. Choose an $x \in V$. We have $V \not\subseteq \text{St}(x, \mathcal{U}')$. Choose a $y \in V \setminus \text{St}(x, \mathcal{U}')$. Then, x and y are $\pi_j^{-1}\mathcal{V}_k$ -near for all $1 \leq k \leq n$ but not \mathcal{U}' -near. By saturation, there exist $x, y \in {}^*X$ such that x and y are ${}^*(\pi_i^{-1}\mathcal{V})$ -near for all $i \in I$ and $\mathcal{V} \in \text{Cov}_u(X_i)$ but not ${}^*\mathcal{U}'$ -near. Hence ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ but $x \not\approx_X y$.
- (2) \Rightarrow (3): Suppose $x \approx_X y$. Let $j \in J$. By Lemma 2.6, we have that ${}^*\pi_{ji}({}^*\pi_j(x)) = {}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y) = {}^*\pi_{ji}({}^*\pi_j(y))$ for all $i \in I$. Hence ${}^*\pi_j(x) \approx_{\mathbf{X}} {}^*\pi_j(y)$. Next, suppose that ${}^*\pi_j(x) \approx_{\mathbf{X}} {}^*\pi_j(y)$ for all $j \in J$.

Fix a $j_0 \in J$. By the definition of $\approx_{\mathbf{X}}$, for any $i \in I$ we have that

$$\begin{aligned} {}^*\pi_i(x) &= {}^*\pi_{j_0 i}({}^*\pi_{j_0}(x)) \\ &\approx_{X_i} {}^*\pi_{j_0 i}({}^*\pi_{j_0}(y)) \\ &= {}^*\pi_i(y). \end{aligned}$$

By (2) we have $x \approx_X y$.

(3) \Rightarrow (4): Suppose that ${}^*\pi_{j_0}(x) \approx_{\mathbf{X}} {}^*\pi_{j_0}(y)$ for some $j_0 \in J$. Let $j \in J$. For any $i \in I$ we have that

$$\begin{aligned} {}^*\pi_{ji}({}^*\pi_j(x)) &= {}^*\pi_i(x) \\ &= {}^*\pi_{j_0 i}({}^*\pi_{j_0}(x)) \\ &\approx_{X_i} {}^*\pi_{j_0 i}({}^*\pi_{j_0}(y)) \\ &= {}^*\pi_i(y) \\ &= {}^*\pi_{ji}({}^*\pi_j(y)). \end{aligned}$$

Hence ${}^*\pi_j(x) \approx_{\mathbf{X}} {}^*\pi_j(y)$. By (3) we have $x \approx_X y$. The converse is clear.

(4) \Rightarrow (2): Suppose that ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ for all $i \in I$. Fix a $j_0 \in J$. We have that ${}^*\pi_{j_0 i}({}^*\pi_{j_0}(x)) = {}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y) = {}^*\pi_{j_0 i}({}^*\pi_{j_0}(y))$ for all $i \in I$. Hence ${}^*\pi_{j_0}(x) \approx_{\mathbf{X}} {}^*\pi_{j_0}(y)$. By (4) we have $x \approx_X y$. The converse immediately follows from Lemma 2.6. □

We can verify that the remaining parts of [8, Chapter 10, 11, 13] can be transferred as well. Thus we obtain the proof of Theorem 4.9.

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